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# Uniqueness and error bounds for eikonal equations with discontinuities (Viscosity Solutions of Differential Equations and Related Topics)

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# Uniqueness and error bounds for eikonal equations with discontinuities

Klaus Deckelnick & Charles M. Elliott

## 1 Introduction

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with a Lipschitz boundary  $\partial\Omega$ . We consider the eikonal equation

$$|\nabla u(x)| = f(x) \quad x \in \Omega \quad (1.1)$$

$$u(x) = \phi(x) \quad x \in \partial\Omega, \quad (1.2)$$

where  $f$  and  $\phi$  are given functions. The equation arises for example in geometric optics, computer vision or robotic navigation. In certain situations it is desirable to allow  $f$  to be discontinuous, e.g. in geometric optics, when light propagates through a layered medium. The aim of this paper is to study the well-posedness of (1.1), (1.2) for right hand sides  $f$  satisfying a one-sided continuity condition (see (2.2) below), that allows certain types of discontinuities. Furthermore, we shall be concerned with an error analysis for a finite difference scheme to approximate the solution of (1.1), (1.2).

The well-posedness of (1.1), (1.2) in the case of continuous  $f$  follows from the theory of viscosity solutions for Hamilton–Jacobi equations  $H(x, u, \nabla u) = 0$  developed in [4]. The notion of viscosity solution was generalised by Ishii [5] to allow for discontinuous Hamiltonians  $H$ . In [11], Tourin proves a comparison result for Hamiltonians, which are allowed to be discontinuous along a smooth surface. Soravia [10] obtains necessary and sufficient conditions for uniqueness of the solution to the boundary value problem. While the work in [11] and [10] is based on Ishii’s notion of solution, several other approaches have been suggested: in [7], Newcomb & Su consider the Dirichlet problem for  $H(\nabla u) = f$  and introduce a notion of solution which they call Monge solution. They obtain a comparison result as well as uniqueness for the Dirichlet problem provided that  $f$  is lower semicontinuous. Ostrov [8] studies an evolutionary Hamilton–Jacobi equation which occurs in the context of radar satellite tracking and obtains a unique solution as the limit of suitable upper and lower solutions. Recently, Camilli & Siconolfi [3] introduced a new notion of solution for Hamilton–Jacobi equations of the form  $H(x, \nabla u) = 0$ , which allows measurable dependence of  $H$  on  $x$  and involves measure-theoretic limits. They prove representation formulae, comparison principles and uniqueness results.

Our work uses Ishii’s definition of solution which we shall recall in §2. For a class of right hand sides  $f$ , which satisfy a suitable one-sided continuity condition we obtain well-posedness of the problem (1.1), (1.2). In §3 we discretize the problem with the help of a finite difference scheme on a regular grid. Under a slightly more restrictive condition on  $f$  we prove that the error between viscosity solution and discrete approximation is of order  $\mathcal{O}(\sqrt{h})$ . We have not included all the details of the proofs of existence and of the error analysis. However, a forthcoming paper, which generalises our approach to Hamilton–Jacobi equations of the form  $H(\nabla u) = f$ , will provide a detailed convergence analysis for a wide class of finite difference schemes as well as numerical tests.

## 2 Existence and Uniqueness

In order to allow for discontinuous functions  $f$  in (1.1) we shall use the following generalisation of the concept of viscosity solution, which was introduced by Ishii in [5].

**Definition 2.1.** A function  $u \in C^0(\bar{\Omega})$  is called a viscosity subsolution (supersolution) of (1.1) if for each  $\zeta \in C^\infty(\Omega)$ : if  $u - \zeta$  has a local maximum (minimum) at a point  $x_0 \in \Omega$ , then

$$|\nabla \zeta(x_0)| \leq f^*(x_0) \quad (\geq f_*(x_0)).$$

Here,

$$f^*(x) := \limsup_{r \rightarrow 0} \{f(y) \mid y \in B_r(x) \cap \Omega\}, \quad f_*(x) := \liminf_{r \rightarrow 0} \{f(y) \mid y \in B_r(x) \cap \Omega\}.$$

A viscosity solution of (1.1), (1.2) then is a function  $u \in C^0(\bar{\Omega})$  which is both a viscosity sub- and supersolution and which satisfies  $u(x) = \phi(x)$  for all  $x \in \partial\Omega$ .

Let us next formulate our assumptions on the data of the problem. We suppose that  $f : \Omega \rightarrow \mathbb{R}$  is Borel measurable and that there exist  $0 < m \leq M < \infty$  such that

$$m \leq f(x) \leq M \quad \forall x \in \Omega. \quad (2.1)$$

Furthermore, we assume that for every  $x \in \Omega$  there exist  $\epsilon_x > 0$  and  $n_x \in S^{n-1}$  so that for all  $y \in \Omega, r > 0$  and all  $d \in S^{n-1}$  with  $|d - n_x| < \epsilon_x$  we have

$$f(y + rd) - f(y) \leq \omega(|y - x| + r), \quad (2.2)$$

where  $\omega : [0, \infty) \rightarrow [0, \infty)$  is continuous, nondecreasing and satisfies  $\omega(0) = 0$ . A similar type of condition was used in [11]; however, in (2.2) it is sufficient to estimate values of  $f$  for vectors whose difference is close to a given direction.

**Example:** Suppose that a surface  $\Gamma$  splits  $\Omega$  into two subdomains  $\Omega_1$  and  $\Omega_2$ , that  $f|_{\Omega_1} \in C^0(\bar{\Omega}_1)$ ,  $f|_{\Omega_2} \in C^0(\bar{\Omega}_2)$  and that

$$\lim_{y \rightarrow x, y \in \Omega_1} f(y) < \lim_{y \rightarrow x, y \in \Omega_2} f(y) \quad \text{for all } x \in \Gamma.$$

In addition, assume that the following uniform cone property holds: for every  $x \in \Gamma$  there exists a neighborhood  $U_x$  and a cone  $C_x$  (which is congruent to a fixed given cone  $C_0$ ) such that  $y \in U_x \cap \bar{\Omega}_1$  implies that  $y + C_x \subset \Omega_1$ . Then (2.2) holds with  $n = n_x$  given by the direction of the cone  $C_x$ .

To see this, observe that the cone condition prevents a situation where  $y \in \bar{\Omega}_1, y + rd \in \Omega_2$ , which would lead to a violation of (2.2) (cf. [11], where  $\Gamma$  is assumed to be smooth).

One can also consider e.g. a two-dimensional domain  $\Omega$ , where three curves of discontinuity meet at a triple junction.

It is not difficult to verify that (2.2) implies

$$f^*(y + rd) - f_*(y) \leq \omega(|y - x| + r) \quad (2.3)$$

for all  $y \in \Omega, r > 0$  and  $d \in S^{n-1}, |d - n_x| < \epsilon_x$ .

Finally, we suppose for simplicity that  $\phi \equiv 0$ .

**Lemma 2.2.** *There exists a viscosity solution  $u \in C^{0,1}(\bar{\Omega})$  of (1.1), (1.2).*

*Proof.* We only sketch the main ideas. Consider the sup-convolution of  $f$ , i.e.

$$f_\epsilon(x) := \sup_{y \in \Omega} \left\{ f(y) - \frac{1}{\epsilon} |x - y|^2 \right\}, \quad x \in \Omega, \epsilon > 0.$$

Clearly,  $f_\epsilon$  is continuous and  $f^*(x) \leq f_\epsilon(x)$  for all  $x \in \Omega$ . Let

$$L_\epsilon(x, y) := \inf \left\{ \int_0^1 f_\epsilon(\gamma(t)) |\gamma'(t)| dt \mid \gamma \in W^{1,\infty}((0, 1); \bar{\Omega}) \text{ with } \gamma(0) = x, \gamma(1) = y \right\}.$$

It is well-known that  $u_\epsilon(x) := \inf_{y \in \partial\Omega} L_\epsilon(x, y)$  is a solution of

$$\begin{aligned} |\nabla u^\epsilon| &= f_\epsilon(x) & x \in \Omega \\ u^\epsilon(x) &= 0 & x \in \partial\Omega \end{aligned}$$

in the viscosity sense. Furthermore, it can be shown that

$$\|u^\epsilon\|_{C^{0,1}(\bar{\Omega})} \leq C(M, \Omega) \quad \text{uniformly in } \epsilon > 0.$$

Thus, there exists a sequence  $(\epsilon_k)_{k \in \mathbb{N}}$  with  $\epsilon_k \searrow 0, k \rightarrow \infty$  and  $u \in C^{0,1}(\bar{\Omega})$  such that  $u^{\epsilon_k} \rightarrow u$  uniformly in  $\bar{\Omega}$  as  $k \rightarrow \infty$ . Using well-known arguments from the theory of viscosity solutions one verifies that  $u$  is a solution of (1.1), (1.2).  $\blacksquare$

Uniqueness of the viscosity solution follows from

**Theorem 2.3.** *Suppose that  $u \in C^0(\bar{\Omega})$  is a subsolution of (1.1),  $v \in C^0(\bar{\Omega})$  is a supersolution of (1.1) and that at least one of the functions belongs to  $C^{0,1}(\bar{\Omega})$ . If  $u \leq v$  on  $\partial\Omega$  then  $u \leq v$  in  $\bar{\Omega}$ .*

*Proof.* Let us assume that  $v \in C^{0,1}(\bar{\Omega})$ . We shall use the approach presented in [6] (see also [11]). Fix  $\theta \in (0, 1)$  and define  $u_\theta(x) := \theta u(x)$ . Next, choose  $x_0 \in \bar{\Omega}$  such that

$$u_\theta(x_0) - v(x_0) = \max_{x \in \bar{\Omega}} (u_\theta(x) - v(x)) =: \mu, \quad (2.4)$$

and suppose that  $\mu > 0$ . Upon replacing  $u, v$  by  $u + k, v + k$ , we may assume that  $u \geq 0$  in  $\bar{\Omega}$ , so that  $u_\theta \leq u$  in  $\bar{\Omega}$ . In particular,  $u_\theta \leq v$  on  $\partial\Omega$ , which implies that  $x_0 \in \Omega$ . Choose  $\epsilon = \epsilon_{x_0}$  and  $n = n_{x_0} \in S^{m-1}$  according to (2.2) and define for  $\lambda > 0, L \geq 1$

$$\Phi(x, y) := u_\theta(x) - v(y) - L\lambda |x - y - \frac{1}{\lambda} n|^2 - |x - x_0|^2, \quad (x, y) \in \bar{\Omega} \times \bar{\Omega}.$$

Choose  $(x_\lambda, y_\lambda) \in \bar{\Omega} \times \bar{\Omega}$  such that

$$\Phi(x_\lambda, y_\lambda) = \max_{(x, y) \in \bar{\Omega} \times \bar{\Omega}} \Phi(x, y).$$

Since  $x_0 \in \Omega$  we also have  $x_0 - \frac{1}{\lambda} n \in \Omega$  for large  $\lambda$ ; using the relation  $\Phi(x_\lambda, y_\lambda) \geq \Phi(x_0, x_0 - \frac{1}{\lambda} n)$  together with (2.4) we infer

$$\begin{aligned} L\lambda |x_\lambda - y_\lambda - \frac{1}{\lambda} n|^2 + |x_\lambda - x_0|^2 &\leq u_\theta(x_\lambda) - v(y_\lambda) - u_\theta(x_0) + v(x_0 - \frac{1}{\lambda} n) \\ &= (u_\theta(x_\lambda) - v(x_\lambda)) - (u_\theta(x_0) - v(x_0)) + v(x_\lambda) - v(y_\lambda) - v(x_0) + v(x_0 - \frac{1}{\lambda} n) \\ &\leq \text{lip}(v) (|x_\lambda - y_\lambda| + \frac{1}{\lambda}) \\ &\leq \text{lip}(v) (|x_\lambda - y_\lambda - \frac{1}{\lambda} n| + \frac{2}{\lambda}). \end{aligned} \quad (2.5)$$

This implies

$$L\lambda|x_\lambda - y_\lambda - \frac{1}{\lambda}n|^2 + |x_\lambda - x_0|^2 \leq \frac{C}{\lambda},$$

where  $C$  depends on  $\text{lip}(v)$  and as a consequence,

$$x_\lambda, y_\lambda \rightarrow x_0, \quad \text{as } \lambda \rightarrow \infty \quad (2.6)$$

$$\lambda|x_\lambda - y_\lambda - \frac{1}{\lambda}n| \leq \frac{C}{\sqrt{L}} < \frac{\epsilon}{2+\epsilon} \quad (2.7)$$

provided that  $L$  is large enough. Since  $u$  is a subsolution, we may deduce from the relation  $\Phi(x_\lambda, y_\lambda) \geq \Phi(x, y_\lambda)$  for  $x \in \bar{\Omega}$  that

$$|2L\lambda(x_\lambda - y_\lambda - \frac{1}{\lambda}n) + 2(x_\lambda - x_0)| \leq \theta f^*(x_\lambda)$$

for large  $\lambda$  and similarly,

$$|2L\lambda(x_\lambda - y_\lambda - \frac{1}{\lambda}n)| \geq f_*(y_\lambda).$$

Combining the above inequalities, we infer

$$(1 - \theta)f^*(y_\lambda) \leq 2|x_\lambda - x_0| + \theta(f^*(x_\lambda) - f_*(y_\lambda)). \quad (2.8)$$

In order to apply (2.2) we write  $x_\lambda = y_\lambda + r_\lambda d_\lambda$ , where

$$d_\lambda = \frac{n + w_\lambda}{|n + w_\lambda|}, \quad r_\lambda = \frac{1}{\lambda} |n + w_\lambda|, \quad w_\lambda = \lambda \left( x_\lambda - y_\lambda - \frac{1}{\lambda}n \right). \quad (2.9)$$

Recalling (2.7) we deduce

$$|d_\lambda - n| \leq \frac{2|w_\lambda|}{1 - |w_\lambda|} \leq \frac{\frac{2\epsilon}{2+\epsilon}}{1 - \frac{\epsilon}{2+\epsilon}} = \epsilon$$

and (2.3) therefore yields

$$f^*(x_\lambda) - f_*(y_\lambda) = f^*(y_\lambda + r_\lambda d_\lambda) - f_*(y_\lambda) \leq \omega(|y_\lambda - x_0| + r_\lambda). \quad (2.10)$$

If we send  $\lambda \rightarrow \infty$  in (2.8) we finally obtain from (2.1), (2.10) and (2.6) that  $(1 - \theta)m \leq 0$ , a contradiction. Thus,  $u_\theta \leq v$  in  $\bar{\Omega}$  and sending  $\theta \nearrow 1$  gives the desired result.  $\blacksquare$

### 3 Numerical scheme and error analysis

Let us assume that  $\Omega = \Pi_{i=1}^n(0, b_i)$  and that the grid size  $h > 0$  is chosen in such a way that  $b_i = N_i h$  for some  $N_i \in \mathbb{N}$ ,  $i = 1, \dots, n$ . We then define

$$\Omega_h := \mathbf{Z}_h^n \cap \Omega, \quad \partial\Omega_h := \mathbf{Z}_h^n \cap \partial\Omega, \quad \bar{\Omega}_h := \Omega_h \cup \partial\Omega_h,$$

where  $\mathbf{Z}_h^n = \{x_\alpha = (h\alpha_1, \dots, h\alpha_n) \mid \alpha_i \in \mathbf{Z}, i = 1, \dots, n\}$ . Our aim is to approximate the viscosity solution  $u$  of (1.1), (1.2) by a grid function  $U : \bar{\Omega}_h \rightarrow \mathbb{R}$  and to prove an estimate for  $\max_{x_\alpha \in \bar{\Omega}_h} |u(x_\alpha) - U(x_\alpha)|$ . Let us abbreviate  $U_\alpha = U(x_\alpha)$  and recall the usual backward and forward difference quotients,

$$D_k^- U_\alpha := \frac{U_\alpha - U_{\alpha - e_k}}{h}, \quad D_k^+ U_\alpha := \frac{U_{\alpha + e_k} - U_\alpha}{h}, \quad x_\alpha \in \Omega_h, \quad k = 1, \dots, n.$$

In order to define the numerical method we introduce the function  $G : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  as

$$G(p_1, q_1, \dots, p_n, q_n) := \left( \sum_{k=1}^n \max(p_k^+, -q_k^-)^2 \right)^{\frac{1}{2}},$$

where  $x^+ = \max(x, 0)$ ,  $x^- = \min(x, 0)$ . The discrete problem now reads: find  $U : \bar{\Omega}_h \rightarrow \mathbb{R}$  such that

$$G(D_1^- U_\alpha, D_1^+ U_\alpha, \dots, D_n^- U_\alpha, D_n^+ U_\alpha) = f(x_\alpha) \quad x_\alpha \in \Omega_h \quad (3.1)$$

$$U_\alpha = 0 \quad x_\alpha \in \partial\Omega_h. \quad (3.2)$$

The above scheme was examined for continuous  $f$  in [9] in the context of shape-from-shading and convergence to the viscosity solution was obtained as a consequence of a result of Barles and Souganidis [2]. In the case of a constant right hand side  $f \equiv 1$ , Zhao [12] recently obtained an  $\mathcal{O}(h)$  error bound. The scheme can be derived by interpreting the viscosity solution  $u$  as the value function of an optimal control problem. For further information and a corresponding list of references we refer to Appendix A (written by M. Falcone) in [1].

The function  $G$  has the following crucial properties:

a) *Consistency*:

$$G(p_1, p_1, \dots, p_n, p_n) = |p| \quad \text{for all } p = (p_1, \dots, p_n) \in \mathbb{R}^n. \quad (3.3)$$

b) *Monotonicity*:

let  $a = (a_1, a_2, \dots, a_{2n-1}, a_{2n})$ ,  $b = (b_1, b_2, \dots, b_{2n-1}, b_{2n}) \in \mathbb{R}^{2n}$  with  $a_k \geq b_k$  for  $k = 1, \dots, 2n$ . Then

$$G(t - a_1, a_2 - t, \dots, t - a_{2n-1}, a_{2n} - t) \leq G(t - b_1, b_2 - t, \dots, t - b_{2n-1}, b_{2n} - t) \quad \forall t \in \mathbb{R}. \quad (3.4)$$

Note that the above properties imply in particular that the solution of (3.1), (3.2) cannot have a local minimum in  $\Omega_h$  and therefore  $U_\alpha \geq 0$  in  $\bar{\Omega}_h$ . In order to carry out our error analysis we need to strengthen (2.2) in that we assume that there exist  $\epsilon > 0$ ,  $K \geq 0$  such that for all  $x \in \Omega$  there is a direction  $n = n_x \in S^{n-1}$  with

$$f(y + rd) - f(y) \leq Kr \quad \forall y \in \Omega, |y - x| < \epsilon \quad \forall d \in S^{n-1}, |d - n| < \epsilon \quad \forall r > 0. \quad (3.5)$$

**Theorem 3.1.** *Let  $u$  be the viscosity solution of (1.1), (1.2) and  $U$  a solution of (3.1), (3.2). Then there exists a constant  $C$ , which is independent of  $h$  such that*

$$\max_{x_\alpha \in \bar{\Omega}_h} |u(x_\alpha) - U(x_\alpha)| \leq C\sqrt{h}.$$

*Proof.* We again only sketch the main ideas. As it seems difficult to use the argument from the uniqueness proof in order to control the maximum of  $u - U$ , we shall resort to the Kruřkov transform. Thus, let  $\tilde{u} : \bar{\Omega} \rightarrow \mathbb{R}$ ,  $\tilde{U} : \bar{\Omega}_h \rightarrow \mathbb{R}$  be defined by

$$\tilde{u}(x) := -e^{-u(x)}, \quad x \in \bar{\Omega}, \quad \tilde{U}_\alpha := -e^{-U_\alpha}, \quad x_\alpha \in \bar{\Omega}_h.$$

One verifies (cf. [4]) that  $\tilde{u}$  is a viscosity solution of

$$f(x)\tilde{u} + |\nabla\tilde{u}| = 0 \quad x \in \Omega \quad (3.6)$$

$$\tilde{u}(x) = -1 \quad x \in \partial\Omega, \quad (3.7)$$

and that  $\tilde{U}$  satisfies

$$f(x_\alpha)\tilde{U}_\alpha + G(D_1^-\tilde{U}_\alpha, D_1^+\tilde{U}_\alpha, \dots, D_n^-\tilde{U}_\alpha, D_n^+\tilde{U}_\alpha) = F_\alpha^h \quad x_\alpha \in \Omega_h \quad (3.8)$$

$$\tilde{U}_\alpha = -1 \quad x_\alpha \in \partial\Omega_h, \quad (3.9)$$

where

$$\max_{x_\alpha \in \Omega_h} |F_\alpha^h| \leq Ch. \quad (3.10)$$

Next, choose  $x_\beta \in \bar{\Omega}_h$  such that

$$|\tilde{u}(x_\beta) - \tilde{U}_\beta| = \max_{x_\alpha \in \bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha|$$

and assume that  $\tilde{u}(x_\beta) \geq \tilde{U}_\beta$ . The opposite case can be treated similarly. If  $\text{dist}(x_\beta, \partial\Omega) \leq \sqrt{h}$ , it follows from (3.7), (3.9) and the Lipschitz continuity of  $\tilde{u}$  that

$$\max_{x_\alpha \in \bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha| = \tilde{u}(x_\beta) - \tilde{U}_\beta \leq C\sqrt{h}.$$

Now suppose that  $\text{dist}(x_\beta, \partial\Omega) > \sqrt{h}$  and define

$$\Phi(x, x_\alpha) := \tilde{u}(x) - \tilde{U}_\alpha - \frac{L_1}{\sqrt{h}} |x - x_\alpha - \sqrt{h}n|^2 - L_2\sqrt{h} |x_\alpha - x_\beta|^2, \quad (x, x_\alpha) \in \bar{\Omega} \times \bar{\Omega}_h.$$

Here,  $n = n_{x_\beta}$  and  $L_1, L_2 \geq 0$  are constants that do not depend on  $h$  and which will be chosen later. There exists  $(x_h, x_{\alpha_h}) \in \bar{\Omega} \times \bar{\Omega}_h$  such that

$$\Phi(x_h, x_{\alpha_h}) = \max_{(x, x_\alpha) \in \bar{\Omega} \times \bar{\Omega}_h} \Phi(x, x_\alpha).$$

Since  $\text{dist}(x_\beta, \partial\Omega) > \sqrt{h}$ , we have  $x_\beta + \sqrt{h}n \in \bar{\Omega}$  and therefore

$$\Phi(x_h, x_{\alpha_h}) \geq \Phi(x_\beta + \sqrt{h}n, x_\beta).$$

From this we infer in a similar way as in (2.5) that

$$|x_{\alpha_h} - x_\beta| < \epsilon, \quad (3.11)$$

$$\frac{1}{\sqrt{h}} |x_h - x_{\alpha_h} - \sqrt{h}n| < \frac{\epsilon}{2 + \epsilon} \quad (3.12)$$

provided that  $L_i = L_i(\text{lip}(\tilde{u}), \epsilon)$ ,  $i = 1, 2$  are sufficiently large ( $\epsilon$  from (3.5)).

Suppose first that  $(x_h, x_{\alpha_h}) \in \Omega \times \Omega_h$ . Since  $\tilde{u}$  is a subsolution of (3.6) we infer

$$f^*(x_h)\tilde{u}(x_h) + \left| \frac{2L_1}{\sqrt{h}} (x_h - x_{\alpha_h} - \sqrt{h}n) \right| \leq 0. \quad (3.13)$$

Keeping the first component of  $\Phi$  fixed we obtain on the other hand for all  $x_\alpha \in \bar{\Omega}_h$

$$\begin{aligned} \tilde{U}_\alpha &\geq \tilde{U}_{\alpha_h} + \frac{L_1}{\sqrt{h}} \left( |x_h - x_{\alpha_h} - \sqrt{h}n|^2 - |x_h - x_\alpha - \sqrt{h}n|^2 \right) \\ &\quad + L_2\sqrt{h} (|x_{\alpha_h} - x_\beta|^2 - |x_\alpha - x_\beta|^2) \\ &=: \tilde{V}_\alpha. \end{aligned}$$

Since  $\tilde{U}_{\alpha_h} = \tilde{V}_{\alpha_h}$ , (3.4) and (3.3) imply

$$\begin{aligned} G(D_1^- \tilde{U}_{\alpha_h}, D_1^+ \tilde{U}_{\alpha_h}, \dots, D_n^- \tilde{U}_{\alpha_h}, D_n^+ \tilde{U}_{\alpha_h}) &\leq G(D_1^- \tilde{V}_{\alpha_h}, D_1^+ \tilde{V}_{\alpha_h}, \dots, D_n^- \tilde{V}_{\alpha_h}, D_n^+ \tilde{V}_{\alpha_h}) \\ &\leq \left| \frac{2L_1}{\sqrt{h}} (x_h - x_{\alpha_h} - \sqrt{h}n) - 2L_2 \sqrt{h} (x_{\alpha_h} - x_\beta) \right| + C\sqrt{h}. \end{aligned}$$

Combining this inequality with (3.8) and (3.10) then yields

$$f(x_{\alpha_h}) \tilde{U}_{\alpha_h} + \left| \frac{2L_1}{\sqrt{h}} (x_h - x_{\alpha_h} - \sqrt{h}n) \right| \geq -|F_{\alpha_h}^h| - C\sqrt{h} \geq -C\sqrt{h}. \quad (3.14)$$

As a result of (3.13), (3.14)

$$\begin{aligned} f(x_{\alpha_h})(\tilde{u}(x_h) - \tilde{U}_{\alpha_h}) &\leq C\sqrt{h} + e^{-u(x_h)}(f^*(x_h) - f(x_{\alpha_h})) \\ &= C\sqrt{h} + e^{-u(x_h)}(f^*(x_{\alpha_h} + r_h d_h) - f(x_{\alpha_h})) \end{aligned} \quad (3.15)$$

where similar to (2.9),  $d_h = \frac{n + w_h}{|n + w_h|}$ ,  $r_h = \sqrt{h}|n + w_h|$ ,  $w_h = \frac{1}{\sqrt{h}}(x_h - x_{\alpha_h} - \sqrt{h}n)$ . Since

$$\begin{aligned} \tilde{u}(x_h) - \tilde{U}_{\alpha_h} &= \Phi(x_h, x_{\alpha_h}) + \frac{L_1}{\sqrt{h}} |x_h - x_{\alpha_h} - \sqrt{h}n|^2 + L_2 \sqrt{h} |x_{\alpha_h} - x_\beta|^2 \\ &\geq \Phi(x_\beta, x_\beta) = \tilde{U}_\beta - \tilde{u}(x_\beta) - L_1 \sqrt{h}, \end{aligned}$$

we finally deduce from (2.1), (3.15) and (3.5) that

$$m(\tilde{u}(x_\beta) - \tilde{U}_\beta) \leq C\sqrt{h} + Kr_h \leq C\sqrt{h}.$$

The cases  $x_h \in \partial\Omega$  or  $x_{\alpha_h} \in \partial\Omega_h$  can be examined with the help of the boundary conditions (3.7), (3.9). Transforming back to  $u$  and  $U$  implies the desired error bound. ■

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